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Restricted gravity: Abelian projection of Einstein's theory

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Summary. — Treating Einstein's theory as a gauge theory of Lorentz group, we decompose the gravitational connection $\mathbf{\Gamma}_\mu$ into the restricted connection made of the potential of the maximal Abelian subgroup H of Lorentz group G and the valence connection made of G/H part of the potential which transforms covariantly under Lorentz gauge transformation. With this we show that Einstein's theory can be decomposed into the restricted gravity made of the restricted connection which has the full Lorentz gauge invariance which has the valence connection as gravitational source. The decomposition shows the existence of a restricted theory of gravitation which has the full general invariance but is much simpler than Einstein's theory. Moreover, it tells that the restricted gravity can be written as an Abelian gauge theory, which implies that the graviton can be described by a massless spin-one field.

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1. – Introduction

Einstein's theory of gravitation is based on the general invariance. But the general invariance is a particular type of gauge invariance, so that Einstein's theory can be understood as a gauge theory [1-3]. For example, it can be viewed as a gauge theory of Lorentz group, where the gauge potential is viewed as the spin connection [4, 5].

It has generally been believed that the gauge principle is so binding that it determines the dynamics uniquely (with the simplicity principle). In fact Einstein's theory is thought to be the simplest theory based on the general invariance. But this is not exactly true. In QCD, for example, we do have the restricted QCD (RCD) which has the full color

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gauge symmetry but simpler than QCD. This is because the non-Abelian gauge theory allows the Abelian decomposition which decomposes the gauge potential to the restricted potential of the maximal Abelian subgroup H of the gauge group G and the valence potential which has the degrees of G/H which transforms covariantly under G [6, 7].

A remarkable feature of this decomposition is that the restricted potential has the full non-Abelian gauge degrees of freedom, in particular the topological degrees of the original gauge theory, in spite of the fact that it is restricted. This means that we can construct a restricted gauge theory, a non-Abelian gauge theory made of only the restricted potential, which nevertheless has the full gauge invariance. Moreover, we can recover the full non-Abelian gauge theory simply by adding the valence part. This tells that the non-Abelian gauge theory can be interpreted as a restricted gauge theory which has the valence potential as the gauge covariant source [6, 7]. The importance of this decomposition is that the restricted gauge theory plays a crucial role in non-Abelian dynamics, in particular to establish the Abelian dominance in color confinement in QCD [8-13].

The main purpose of this paper is to discuss a similar Abelian decomposition of Einstein's theory. Regarding the theory as a gauge theory of Lorentz group and applying the Abelian decomposition to the gauge potential of Lorentz group, we first show that we can decompose the gravitational connection to the restricted connection and the valence connection. With this we decompose the Einstein's theory into the restricted part made of the restricted connection and the valence part made of the gauge covariant valence connection. We show that Einstein's theory allows two different Abelian decompositions, light-like decomposition and non–light-like decomposition. This is because the Lorentz group has two maximal Abelian subgroups.

Our analysis shows that Einstein's theory can be viewed as a restricted theory of gravitation which has the gauge covariant valence connection as the gravitational source. Moreover, our analysis shows that the restricted gravity which describes the core dynamics of Einstein's theory can be put into an Abelian gauge theory which is able to describe the gravitational plane wave. This implies the Abelian dominance in Einstein's theory. In particular, this implies that the graviton can be described by a massless spin-one gauge potential.

2. – Abelian decomposition of $SU(2)$: A review

To understand how the Abelian decomposition works in Einstein's theory, it is important to understand the Abelian decomposition $SU(2)$ gauge theory first. Let $\hat{n}$ be an arbitrary isotriplet unit vector field of $SU(2)$, and identify the maximal Abelian subgroup to be the $U(1)$ subgroup which leaves $\hat{n}$ invariant. Clearly $\hat{n}$ selects the "Abelian" direction (*i.e.*, the color charge direction) at each space-time point, and the the Abelian magnetic isometry can be described by the following constraint equation:

$$D_\mu \hat{n} = \partial_\mu \hat{n} + g\vec{A}_\mu \times \hat{n} = 0, \quad (\hat{n}^2 = 1). \tag{1}$$

This has the unique solution for $\vec{A}_\mu$ which defines the restricted potential $\hat{A}_\mu$ which leaves $\hat{n}$ invariant under the parallel transport,

$$\hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n}, \tag{2}$$

where $A_\mu = \hat{n} \cdot \vec{A}_\mu$ is the "electric" potential. This process of selecting the restricted potential is called the Abelian projection [6,7].

With the Abelian projection we can retrieve the full gauge potential by adding the gauge covariant valence potential $\vec{X}_\mu$ to the restricted potential,

$$\vec{A}_\mu = A_\mu \hat{n} - \frac{1}{g}\hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu, \quad (\hat{n}^2 = 1,\ \hat{n} \cdot \vec{X}_\mu = 0). \tag{3}$$

This is the Abelian decomposition which decomposes the gauge potential into the restricted potential $\hat{A}_\mu$ and the valence potential $\vec{X}_\mu$ [6,7].

Let $\vec{\alpha}$ is an infinitesimal gauge parameter. Under the infinitesimal gauge transformation

$$\delta \hat{n} = -\vec{\alpha} \times \hat{n}, \qquad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \tag{4}$$

one has

$$\delta A_\mu = \frac{1}{g}\hat{n} \cdot \partial_\mu \vec{\alpha}, \qquad \delta \hat{A}_\mu = \frac{1}{g}\hat{D}_\mu \vec{\alpha}, \qquad \delta \vec{X}_\mu = -\vec{\alpha} \times \vec{X}_\mu. \tag{5}$$

This shows that $\hat{A}_\mu$ by itself describes an $SU(2)$ connection which enjoys the full $SU(2)$ gauge degrees of freedom. Furthermore $\vec{X}_\mu$ transforms covariantly under the gauge transformation. Most importantly, the decomposition is gauge-independent. Once the color direction $\hat{n}$ is selected, the decomposition follows independent of the choice of a gauge.

The restricted potential $\hat{A}_\mu$ actually has a dual structure. To see this let $\hat{n}_i (i = 1, 2, 3)$ with $\hat{n}_3 = \hat{n}$ be a right-handed orthonormal $SU(2)$ basis, and notice that

$$\begin{aligned} &\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g\hat{A}_\mu \times \hat{A}_\nu = (F_{\mu\nu} + H_{\mu\nu})\hat{n}, \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad H_{\mu\nu} = -\frac{1}{g}\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \widetilde{C}_\nu - \partial_\nu \widetilde{C}_\mu, \end{aligned} \tag{6}$$

where $\widetilde{C}_\mu = \frac{1}{g}\hat{n}_1 \cdot \partial_\mu \hat{n}_2$ is the "magnetic" potential which describes the non-Abelian monopole [6,7]. This confirms that $\hat{A}_\mu$ has a dual structure.

As importantly $\hat{A}_\mu$ retains the essential topological characteristics of the original non-Abelian potential, because it has the full non-Abelian gauge degrees of freedom. In particular, $\hat{n}$ represents the monopole topology $\pi_2(S^2)$ which describes the mapping from S^2 in 3-dimensional space R^3 to the coset space $SU(2)/U(1)$ [6,14]. Moreover, it describes the vacuum topology $\pi_3(S^3) \simeq \pi_3(S^2)$ which describes the mapping from the compactified 3-dimensional space S^3 to the group space S^3 [15-17].

With (3) we have

$$\vec{F}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu + g\vec{X}_\mu \times \vec{X}_\nu, \tag{7}$$

so that the Yang-Mills Lagrangian is expressed as

$$\mathcal{L} = -\frac{1}{4}\hat{F}^2_{\mu\nu} - \frac{1}{4}(\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu)^2 - \frac{g}{2}\hat{F}_{\mu\nu} \cdot (\vec{X}_\mu \times \vec{X}_\nu) - \frac{g^2}{4}(\vec{X}_\mu \times \vec{X}_\nu)^2. \tag{8}$$

This tells that the non-Abelian gauge theory can be viewed as the restricted gauge theory made of the restricted potential, which has the gauge covariant valence potential as an additional source.

The Abelian decomposition can play the crucial role to cure this defect, because it can separate the monopole potential gauge independently. Indeed, using the Abelian decomposition, KEK lattice group has recently demonstrated that it is the monopole condensation (the $H_{\mu\nu}$ condensation), not the magnetic condensation (the $F_{\mu\nu}$ condensation), which is responsible for the color confinement in QCD [12, 13]. This is really remarkable, because this establishes the dual Meissner effect based on the monopole condensation as a gauge independent phenomenon in QCD. This would have been impossible without the Abelian decomposition. The Abelian decomposition of Einstein's theory could play an equally important role in gravity.

3. – Abelian decomposition of gravitational connection

We can apply the above Abelian decomposition to Einstein's theory, regarding Einstein's theory as a gauge theory of the Lorentz group $SO(3,1)$. To do this we introduce a coordinate basis and an orthonormal basis

$$\begin{aligned} &\partial_\mu = e_\mu^{\ a}\xi_a, \quad (\mu,\nu = t,x,y,z), \qquad \xi_a = e_a^{\ \mu}\partial_\mu, \quad (a,b=0,1,2,3), \\ &[\partial_\mu,\partial_\nu]=0, \qquad [\xi_a,\xi_b] = f_{ab}^{\ \ c}\xi_c, \qquad f_{ab}^{\ \ c} = (e_a^{\ \mu}\partial_\mu e_b^{\ \nu} - e_b^{\ \mu}\partial_\mu e_a^{\ \nu})e_\nu^{\ c}, \end{aligned} \tag{9}$$

where $e_\mu^{\ a}$ and $e_a^{\ \mu}$ are the tetrad and inverse tetrad. Let $J_{ab} = -J_{ba}$ be the generators of the Lorentz group,

$$\begin{aligned} &[J_{ab},J_{cd}] = \eta_{ac}J_{bd} - \eta_{bc}J_{ad} + \eta_{bd}J_{ac} - \eta_{ad}J_{bc} = f_{ab,cd}^{\quad\ \ mn}J_{mn}, \\ &f_{ab,cd}^{\quad\ \ mn} = \eta_{ac}\delta_b^{\ [m}\delta_d^{\ n]} - \eta_{bc}\delta_a^{\ [m}\delta_d^{\ n]} + \eta_{bd}\delta_a^{\ [m}\delta_c^{\ n]} - \eta_{ad}\delta_b^{\ [m}\delta_c^{\ n]}, \end{aligned} \tag{10}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric. Instead of $(ab,cd,\ldots)$ we can use the index $(A,B,\ldots) = (1,2,3,4,5,6) = (23,31,12,01,02,03)$, and express this as

$$\begin{aligned} &[L_i,L_j] = \epsilon_{ijk}L_k, \qquad [L_i,K_j] = \epsilon_{ijk}K_k, \qquad [K_i,K_j] = -\epsilon_{ijk}L_k, \quad (i,j,k=1,2,3), \\ &L_{1,2,3} = J_{23,31,12}, \qquad K_{1,2,3} = J_{01,02,03}, \end{aligned}$$

where L_i and K_i are the 3-dimensional rotation and boost generators.

When we regard Einstein's theory as a gauge theory of the Lorentz group, the gravitational connection $\Gamma_{\mu\nu}^{\ \ \rho}$ corresponds to the gauge potential $\Gamma_\mu^{\ ab}$, and the curvature tensor $R_{\mu\nu}^{\ \ ab}$ corresponds to the gauge field strength $F_{\mu\nu}^{\ \ ab}$ of the Lorentz group. And to obtain the desired decomposition we have to decompose the gauge potential $\Gamma_\mu^{\ ab}$ first. To apply the above $SU(2)$ decomposition to the Lorentz group, however, we have to keep in mind the followings. First, the Lorentz group is non-compact, so that the invariant metric is indefinite. Second, the Lorentz group has the well-known invariant tensor ϵ_{abcd} which allows the dual transformation. Third, the Lorentz group has rank two, so that it has two commuting Abelian subgroups and two Casimir invariants. Finally, the Lorentz group has two different maximal Abelian subgroups A_2 and B_2 [18].

The invariant metric δ_{AB} of Lorentz group is given by

$$\delta_{AB} = -\frac{1}{4}f_{AC}^{\ \ \ D}f_{BD}^{\ \ \ C} = \mathrm{diag}(+1,+1,+1,-1,-1,-1). \tag{11}$$

Let $p^{ab}(p^{ab} = -p^{ba})$ (or p^A) be a gauge covariant sextet vector which forms an adjoint representation of Lorentz group. Clearly p^{ab} can be understood as an antisymmetric tensor in 4-dimensional Minkowski space which can be expressed by two 3-dimensional vectors $\vec{m}$ and $\vec{e}$, which transform exactly like the magnetic and electric components of an electromagnetic tensor under the 4-dimensional Lorentz transformation. And we denote p^{ab} by $\mathbf{p}$,

$$\mathbf{p} = \frac{1}{2} p_{ab} \mathbf{I}^{ab} = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix}, \qquad p^{ab} = \mathbf{p} \cdot \mathbf{I}^{ab} = \frac{1}{2} p^{mn} I_{mn}{}^{ab}, \qquad \mathbf{I}^{ab} = \begin{pmatrix} \hat{m}^{ab} \\ \hat{e}^{ab} \end{pmatrix},$$
$$\hat{m}_i{}^{ab} = \epsilon_{0i}{}^{ab}, \qquad \hat{e}_i{}^{ab} = \left(\delta_0{}^a \delta_i{}^b - \delta_0{}^b \delta_i{}^a\right),$$
$$I_{mn}{}^{ab} = \left(\delta_m^a \delta_n{}^b - \delta_m^b \delta_n{}^a\right) = -(J_{mn})^{ab}, \tag{12}$$

where $m_i = \epsilon_{ijk} p^{jk}/2$ $(i, j, k = 1, 2, 3)$ is the magnetic (or rotation) part and $e_i = p^{0i}$ is the electric (or boost) part of $\mathbf{p}$. From the invariant metric (11) we have

$$\mathbf{p}^2 = \frac{1}{2} p_{ab} p^{ab} = \vec{m}^2 - \vec{e}^2, \tag{13}$$

so that the invariant length can be positive, zero, or negative. This, of course, is due to the fact that the invariant metric (11) is indefinite.

The Lorentz group has another important invariant tensor $\epsilon_{AB} = \epsilon_{abcd}$, so that any adjoint representation of Lorentz group has its dual partner. In particular, $\mathbf{p}$ has the dual vector $\tilde{\mathbf{p}}$ defined by $\widetilde{p}^{ab} = \epsilon^{abcd} p_{cd}/2$. With (12) we have (with $\epsilon_{0123} = +1$)

$$\tilde{\mathbf{p}} = \begin{pmatrix} \vec{e} \\ -\vec{m} \end{pmatrix}, \qquad \tilde{\tilde{\mathbf{p}}} = -\mathbf{p}, \qquad \tilde{\mathbf{p}}^2 = -\mathbf{p}^2, \qquad \mathbf{p} \cdot \tilde{\mathbf{p}} = 2\vec{m} \cdot \vec{e},$$
$$[p, \widetilde{p}] = 0, \qquad \mathbf{p} \times \tilde{\mathbf{p}} = 0. \tag{14}$$

This tells that any two vectors which are dual to each other are always commuting.

Let $(\hat{n}_1, \hat{n}_2, \hat{n}_3 = \hat{n})$ be 3-dimensional unit vectors $(\hat{n}_i^2 = 1)$ which form a right-handed orthonormal basis with $\hat{n}_1 \times \hat{n}_2 = \hat{n}_3$, and let

$$\mathbf{l}_i = \begin{pmatrix} \hat{n}_i \\ 0 \end{pmatrix}, \qquad \mathbf{k}_i = \begin{pmatrix} 0 \\ \hat{n}_i \end{pmatrix} = -\tilde{\mathbf{l}}_i. \tag{15}$$

Clearly we have

$$\mathbf{l}_i \cdot \mathbf{l}_j = \delta_{ij}, \qquad \mathbf{l}_i \cdot \mathbf{k}_j = 0, \qquad \mathbf{k}_i \cdot \mathbf{k}_j = -\delta_{ij},$$
$$\mathbf{l}_i \times \mathbf{l}_j = \epsilon_{ijk} \mathbf{l}_k, \qquad \mathbf{l}_i \times \mathbf{k}_j = \epsilon_{ijk} \mathbf{k}_k, \qquad \mathbf{k}_i \times \mathbf{k}_j = -\epsilon_{ijk} \mathbf{l}_k, \tag{16}$$

so that $(\mathbf{l}_i, \mathbf{k}_i)$, or equivalently $(\mathbf{l}_i, \tilde{\mathbf{l}}_i)$, forms an orthonormal basis of the adjoint representation of the Lorentz group.

To make the desired Abelian decomposition we have to choose the gauge covariant sextet vector fields which form adjoint representation of the Lorentz group which describe the desired magnetic isometry. To see what types of isometry is possible, it is important to remember that the Lorentz group has two 2-dimensional maximal Abelian subgroups,

A_2 whose generators are made of L_3 and K_3 and B_2 whose generators are made of $(L_1 + K_2)/\sqrt{2}$ and $(L_2 - K_1)/\sqrt{2}$ [18].

This tells that we have two possible Abelian decompositions of the gravitational connection. And in both cases the magnetic isometry is described by two, not one, commuting sextet vector fields of the Lorentz group which are dual to each other. To see this let us denote one of the isometry vector field by $\mathbf{p}$ which satisfy the isometry condition

$$D_\mu \mathbf{p} = (\partial_\mu + \mathbf{\Gamma}_\mu \times)\mathbf{p} = 0, \tag{17}$$

where we have normalized the coupling constant to be the unit (which one can always do without loss of generality). Now, notice that the above condition automatically assures

$$D_\mu \tilde{\mathbf{p}} = (\partial_\mu + \mathbf{\Gamma}_\mu \times)\tilde{\mathbf{p}} = 0, \tag{18}$$

because ϵ_{abcd} is an invariant tensor. This tells that when $\mathbf{p}$ is an isometry, $\tilde{\mathbf{p}}$ also becomes an isometry.

Since the Lorentz group has two invariant tensors it has two Casimir invariants. And it is useful to characterize the isometry by two Casimir invariants. Let the isometry be described by $\mathbf{p}$ and $\tilde{\mathbf{p}}$. It has two Casimir invariants α and β,

$$\alpha = \mathbf{p} \cdot \mathbf{p} = \vec{m}^2 - \vec{e}^2, \qquad \beta = \mathbf{p} \cdot \tilde{\mathbf{p}} = 2\vec{m} \cdot \vec{e}. \tag{19}$$

But we can always choose (α, β) to be $(\pm 1, 0)$ or $(0, 0)$ unless $\alpha^2 + \beta^2 = 0$. Physically this means that the magnetic isometry in Einstein's theory can be classified by the non–light-like (or rotation/boost) isometry and the light-like (or null) isometry whose Casimir invariants are denoted by $(\pm 1, 0)$ and $(0, 0)$, respectively.

3˙1. *A_2 (Non–light-like) isometry.* – Let the maximal Abelian subgroup be A_2. In this case the isometry is made of L_3 and K_3, and we have two sextet vector fields which describes the isometry which are dual to each other. Let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ be the two isometry vector fields which correspond to L_3 and K_3. Clearly we can put

$$\mathbf{p} = f\mathbf{l}_3 = f \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \qquad \tilde{\mathbf{p}} = f\tilde{\mathbf{l}}_3 = f \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix}, \tag{20}$$

where f is an arbitrary function of space-time. The Casimir invariants of the isometry vectors are given by $(f^2, 0)$. But just as in $SU(2)$ gauge theory the isometry condition (17) requires f to be a constant, because

$$\partial_\mu f^2 = \partial_\mu \mathbf{p}^2 = D_\mu \mathbf{p}^2 = 2\mathbf{p} \cdot D_\mu \mathbf{p} = 0. \tag{21}$$

And we can always normalize $f = 1$ without loss of generality.

So the A_2 isometry can always be written as

$$\mathbf{l} = \mathbf{l}_3 = \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \qquad \tilde{\mathbf{l}} = \tilde{\mathbf{l}}_3 = \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix}, \tag{22}$$
$$D_\mu \mathbf{l} = 0, \qquad D_\mu \tilde{\mathbf{l}} = 0,$$

whose Casimir invariants are fixed by $(1,0)$. With this we find the restricted connection $\hat{\mathbf{\Gamma}}_\mu$ which satisfies the isometry condition

$$\hat{\mathbf{\Gamma}}_\mu = A_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}} - \mathbf{l} \times \partial_\mu \mathbf{l}, \qquad A_\mu = \mathbf{l} \cdot \mathbf{\Gamma}_\mu, \qquad B_\mu = \tilde{\mathbf{l}} \cdot \mathbf{\Gamma}_\mu, \tag{23}$$

where A_μ and B_μ are two Abelian connections of $\mathbf{l}$ and $\tilde{\mathbf{l}}$ components which are not restricted by the isometry condition. The restricted field strength $\hat{\mathbf{R}}_{\mu\nu}$ is given by

$$\begin{aligned} \hat{\mathbf{R}}_{\mu\nu} &= \partial_\mu \hat{\mathbf{\Gamma}}_\nu - \partial_\nu \hat{\mathbf{\Gamma}}_\mu + \hat{\mathbf{\Gamma}}_\mu \times \hat{\mathbf{\Gamma}}_\nu = \bar{A}_{\mu\nu} \mathbf{l} - B_{\mu\nu} \tilde{\mathbf{l}}, \\ \bar{A}_{\mu\nu} &= A_{\mu\nu} + H_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu, \qquad \bar{A}_\mu = A_\mu + \widetilde{C}_\mu, \\ A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu} = -\mathbf{l} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}) = \partial_\mu \widetilde{C}_\nu - \partial_\nu \widetilde{C}_\mu, \quad \widetilde{C}_\mu = \hat{n}_1 \cdot \partial_\mu \hat{n}_2, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu. \end{aligned} \tag{24}$$

Notice that $\widetilde{C}_\mu$ and $H_{\mu\nu}$ are formally identical to the magnetic potential field strength of the $SU(2)$ gauge theory. This tells that the topology of this isometry is identical to that of the $SU(2)$ subgroup.

With this the full connection of Lorentz group is given by

$$\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu, \qquad \mathbf{l} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{Z}_\mu = 0, \tag{25}$$

where $\mathbf{Z}_\mu$ is the valence connection which transforms covariantly under the Lorentz gauge transformation. The corresponding field strength $\mathbf{R}_{\mu\nu}$ which describes the curvature tensor is written as

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + \mathbf{\Gamma}_\mu \times \mathbf{\Gamma}_\nu = \hat{\mathbf{R}}_{\mu\nu} + \mathbf{Z}_{\mu\nu}, \\ \mathbf{Z}_{\mu\nu} &= \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu, \qquad \hat{D}_\mu = \partial_\mu + \hat{\mathbf{\Gamma}}_\mu \times, \end{aligned} \tag{26}$$

where $\mathbf{Z}_{\mu\nu}$ is the valence part of the curvature tensor.

3˙2. B_2 *(Light-like) isometry.* – This is when the isometry group is made of $(L_1+K_2)/\sqrt{2}$ and $(L_2-K_1)/\sqrt{2}$. Let $\mathbf{p}$ and $\tilde{\mathbf{p}}$ be the two isometry vector fields which correspond to $(L_1+K_2)/\sqrt{2}$ and $(L_2-K_1)/\sqrt{2}$ which are dual to each other. In this case we can write

$$\mathbf{p} = f\Big(\frac{\mathbf{l}_1 + \mathbf{k}_2}{\sqrt{2}}\Big) = \frac{f}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \qquad \tilde{\mathbf{p}} = f\Big(\frac{\mathbf{l}_2 - \mathbf{k}_1}{\sqrt{2}}\Big) = \frac{f}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}. \tag{27}$$

But notice that the Casimir invariants (α,β) of the isometry vectors are given by $(0,0)$ independent of f. Moreover, here (unlike the A_2 case) the isometry condition does not restrict f at all, because we have $\mathbf{p}^2 = 0$ independent of f.

Let us put $f = e^\lambda$ and express the B_2 isometry by

$$\begin{aligned} \mathbf{j} &= \frac{e^\lambda}{\sqrt{2}} (\mathbf{l}_1 + \mathbf{k}_2) = \frac{e^\lambda}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \qquad \tilde{\mathbf{j}} = \frac{e^\lambda}{\sqrt{2}} (\mathbf{l}_2 - \mathbf{k}_1) = \frac{e^\lambda}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}, \\ &D_\mu \mathbf{j} = 0, \qquad D_\mu \tilde{\mathbf{j}} = 0. \end{aligned} \tag{28}$$

To find the restricted connection $\hat{\mathbf{\Gamma}}$ which satisfies the isometry condition we introduce 4 more basis vectors which together with $\mathbf{j}$ and $\tilde{\mathbf{j}}$ form a complete basis

$$\mathbf{k} = \frac{e^{-\lambda}}{\sqrt{2}}(\mathbf{l}_1 - \mathbf{k}_2), \qquad \tilde{\mathbf{k}} = -\frac{e^{-\lambda}}{\sqrt{2}}(\mathbf{l}_2 + \mathbf{k}_1), \quad \mathbf{l} = -\mathbf{j} \times \tilde{\mathbf{k}}, \quad \tilde{\mathbf{l}} = \mathbf{j} \times \mathbf{k}. \tag{29}$$

With this we find the following restricted connection for the B_2 isometry:

$$\hat{\mathbf{\Gamma}}_\mu = \Gamma_\mu \mathbf{j} - \widetilde{\Gamma}_\mu \tilde{\mathbf{j}} - \frac{1}{2}(\mathbf{k} \times \partial_\mu \mathbf{j} - \tilde{\mathbf{k}} \times \partial_\mu \tilde{\mathbf{j}}), \quad \Gamma_\mu = \mathbf{k} \cdot \mathbf{\Gamma}_\mu, \quad \widetilde{\Gamma}_\mu = \tilde{\mathbf{k}} \cdot \mathbf{\Gamma}_\mu, \tag{30}$$

where Γ_μ and $\widetilde{\Gamma}_\mu$ are two Abelian connections of $\mathbf{j}$ and $\tilde{\mathbf{j}}$ components which are not restricted by the isometry condition.

The restricted curvature tensor $\hat{\mathbf{R}}_{\mu\nu}$ is given by

$$\begin{aligned}
&\hat{\mathbf{R}}_{\mu\nu} = \partial_\mu \hat{\mathbf{\Gamma}}_\nu - \partial_\nu \hat{\mathbf{\Gamma}}_\mu + \hat{\mathbf{\Gamma}}_\mu \times \hat{\mathbf{\Gamma}}_\nu = K_{\mu\nu}\mathbf{j} - \widetilde{K}_{\mu\nu}\tilde{\mathbf{j}}, \\
&K_{\mu\nu} = \Gamma_{\mu\nu} + H_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu, \quad \widetilde{K}_{\mu\nu} = \widetilde{\Gamma}_{\mu\nu} + \widetilde{H}_{\mu\nu} = \partial_\mu \widetilde{K}_\nu - \partial_\nu \widetilde{K}_\mu, \\
&\Gamma_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu, \quad \widetilde{\Gamma}_{\mu\nu} = \partial_\mu \widetilde{\Gamma}_\nu - \partial_\nu \widetilde{\Gamma}_\mu, \\
&H_{\mu\nu} = -\mathbf{k} \cdot (\partial_\mu \mathbf{j} \times \partial_\nu \mathbf{k} - \partial_\nu \mathbf{j} \times \partial_\mu \mathbf{k}) = \partial_\mu \widetilde{C}^1_\nu - \partial_\nu \widetilde{C}^1_\mu, \quad \widetilde{C}^1_\mu = \frac{e^{-\lambda}}{\sqrt{2}} \vec{n}_2 \cdot \partial_\mu \vec{n}_3, \\
&\widetilde{H}_{\mu\nu} = -\tilde{\mathbf{k}} \cdot (\partial_\mu \mathbf{j} \times \partial_\nu \mathbf{k} - \partial_\nu \mathbf{j} \times \partial_\mu \mathbf{k}) = \partial_\mu \widetilde{C}^2_\nu - \partial_\nu \widetilde{C}^2_\mu, \quad \widetilde{C}^2_\mu = \frac{e^{-\lambda}}{\sqrt{2}} \vec{n}_1 \cdot \partial_\mu \vec{n}_3, \\
&K_\mu = \Gamma_\mu + \widetilde{C}^1_\mu, \qquad \widetilde{K}_\mu = \widetilde{\Gamma}_\mu + \widetilde{C}^2_\mu.
\end{aligned} \tag{31}$$

Notice that $\hat{\mathbf{R}}_{\mu\nu}$ is orthogonal to $\mathbf{l}$ and $\tilde{\mathbf{l}}$. This should be contrasted with the restricted curvature tensor (24) of the A_2 isometry.

With this we obtain the full gauge potential of Lorentz group by adding the valence connection $\mathbf{Z}_\mu$,

$$\mathbf{\Gamma}_\mu = \hat{\mathbf{\Gamma}}_\mu + \mathbf{Z}_\mu, \qquad \mathbf{k} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{k}} \cdot \mathbf{Z}_\mu = 0. \tag{32}$$

With this we have the full-curvature tensor

$$\begin{aligned}
&\mathbf{R}_{\mu\nu} = \partial_\mu \mathbf{\Gamma}_\nu - \partial_\nu \mathbf{\Gamma}_\mu + \mathbf{\Gamma}_\mu \times \mathbf{\Gamma}_\nu = \hat{\mathbf{R}}_{\mu\nu} + \mathbf{Z}_{\mu\nu}, \\
&\mathbf{Z}_{\mu\nu} = \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu, \qquad \hat{D}_\mu = \partial_\mu + \hat{\mathbf{\Gamma}}_\mu \times .
\end{aligned} \tag{33}$$

This completes the B_2 decomposition of the gravitational connection.

The above result tells that there exist two different Abelian decompositions of the gravitational connection and the curvature tensor which decompose them into the restricted part and the valence part. This allows us to decompose the Einstein's theory in terms of the restricted part and the valence part.

4. – Abelian projection of einstein's theory: restricted gravity

In the absence of the matter field, the Einstein-Hilbert action in the first-order formalism is given by

$$
\begin{aligned}
(34) \quad S[e_a^\mu, \mathbf{\Gamma}_\mu] &= \frac{1}{16\pi G_N}\int \Big(e e_a^\mu e_b^\nu \mathbf{I}^{ab}\cdot \mathbf{R}_{\mu\nu}\Big)\mathrm{d}^4x = \frac{1}{16\pi G_N}\int \Big(\mathbf{g}_{\mu\nu}\cdot \mathbf{R}^{\mu\nu}\Big)\mathrm{d}^4x,\\
\mathbf{g}_{\mu\nu} &= e e_\mu^a e_\nu^b \mathbf{I}_{ab}, \qquad I_{ab}{}^{cd} = (\delta_a{}^c\delta_b{}^d - \delta_a{}^d\delta_b{}^c),\\
g_{\mu\nu}{}^{ab} &= e(e_\mu{}^a e_\nu{}^b - e_\nu{}^b e_\mu{}^a), \quad e = \mathrm{Det}(e_{a\mu}).
\end{aligned}
$$

Notice that here we have introduced the Lorentz covariant four index metric tensor $\mathbf{g}_{\mu\nu}$ (or equivalently $\mathbf{I}_{ab}$) which forms an adjoint representation of Lorentz group. Since $\mathbf{g}_{ab} = e_a^\mu e_b^\nu \mathbf{g}_{\mu\nu} = e\mathbf{I}_{ab}$, $\mathbf{I}_{ab}$ can (up to the scale factor e) be viewed as the "Minkowskian" four index metric tensor $\mathbf{g}_{\mu\nu}$ expressed in the orthonormal Lorentz frame. From (34) we have the following equation of motion:

$$
\begin{aligned}
(35) \qquad \delta e_{\mu a}; &\qquad \mathbf{g}_{\mu\nu}\cdot \mathbf{R}^{\nu\rho} e_{\rho a} = R_{\mu a} = 0,\\
\delta \mathbf{\Gamma}_\mu; &\qquad \mathscr{D}_\mu \mathbf{g}^{\mu\nu} = (\nabla_\mu + \mathbf{\Gamma}_\mu\times)\mathbf{g}^{\mu\nu} = 0,
\end{aligned}
$$

where $R_{\mu a} = e^{\nu b}R_{\mu\nu ab}$ is the Ricci tensor. The first equation assures that, in the absence of matter fields, the Ricci tensor must vanish. The second equation tells that the gauge potential of Lorentz group $\Gamma_\mu{}^{ab}$ must be the matric compatible spin connection $\omega_\mu{}^{ab}$,

$$
(36) \qquad \Gamma_\mu{}^{ab} = \frac{1}{2}(e^{a\nu}e_{c\mu}\partial^b e^c{}_\nu + e^{a\nu}\partial_\mu e^b{}_\nu + \partial^b e^a{}_\mu - e^{b\nu}e_{c\mu}\partial^a e^c{}_\nu - e^{b\nu}\partial_\mu e^a{}_\nu - \partial^a e^b{}_\mu).
$$

Notice that the second equation of (35) is nothing but the metric-compatibility condition of the connection

$$
(37) \qquad \mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0 \Longleftrightarrow \nabla_\alpha g_{\mu\nu} = 0.
$$

Actually in the Lorentz gauge formalism of Einstein's theory we automatically have this metric-compatibility, because we already have $D_\mu \eta_{ab} = 0$. Indeed, with this and with the identity $\mathscr{D}_\mu e_\nu{}^a = \partial_\mu e_\nu{}^a - \Gamma_{\mu\nu}{}^\alpha e_\alpha{}^a + \Gamma_\mu{}^a{}_b e_\nu{}^b = 0$, we can reduce $\mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0$ to

$$
(38) \qquad D_\mu \mathbf{I}^{ab} = 0,
$$

which becomes an identity. So the second equation of (35) can actually be viewed as an identity.

Now, just as in the $SU(2)$ gauge theory, we can obtain the restricted theory of gravity which has the full Lorentz gauge invariance excluding the valence connection $\mathbf{Z}_\mu$. This is because the valence connection is gauge covariant. And here again we have two types of restricted gravity, the A_2 gravity and the B_2 gravity. We discuss them separately.

4$\dot{}$1. A_2 *Gravity.* – Let $\mathbf{Z}_\mu = 0$ and let

$$
\begin{aligned}
(39)\qquad & \mathbf{g}_{\mu\nu} = \hat{\mathbf{g}}_{\mu\nu} + \mathbf{G}_{\mu\nu}, \\
& \hat{\mathbf{g}}_{\mu\nu} = G_{\mu\nu}\mathbf{l} - \widetilde{G}_{\mu\nu}\tilde{\mathbf{l}}, \qquad \mathbf{G}_{\mu\nu} = G^1_{\mu\nu}\mathbf{l}_1 - \widetilde{G}^1_{\mu\nu}\tilde{\mathbf{l}}_1 + G^2_{\mu\nu}\mathbf{l}_2 - \widetilde{G}^2_{\mu\nu}\tilde{\mathbf{l}}_2, \\
& G_{\mu\nu} = e^a_\mu e^b_\nu l_{ab}, \qquad G^1_{\mu\nu} = e^a_\mu e^b_\nu l^1_{ab}, \qquad G^2_{\mu\nu} = e^a_\mu e^b_\nu l^2_{ab}, \\
& \widetilde{G}_{\mu\nu} = e^a_\mu e^b_\nu \tilde{l}_{ab}, \qquad \widetilde{G}^1_{\mu\nu} = e^a_\mu e^b_\nu \tilde{l}^1_{ab}, \qquad \widetilde{G}^2_{\mu\nu} = e^a_\mu e^b_\nu \tilde{l}^2_{ab}.
\end{aligned}
$$

In this case (35) is reduced to

$$
(40)\qquad
\begin{gathered}
G_{\mu\nu}(\partial^\nu \bar{A}^\rho - \partial^\rho \bar{A}^\nu) - \widetilde{G}_{\mu\nu}(\partial^\nu B^\rho - \partial^\rho B^\nu) = 0, \\
\nabla_\mu G^{\mu\nu} = 0, \qquad \nabla_\mu \widetilde{G}^{\mu\nu} = 0, \\
\hat{\mathscr{D}}_\mu \mathbf{G}^{\mu\nu} = 0.
\end{gathered}
$$

This provides the equations of motion for the A_2 gravity.

To understand the physics behind (40) notice that the first and last equations are the first order differential equations, so that they do not describe the dynamical (*i.e.*, propagating) graviton. They are the constraint equations which determine the connection in terms of the metric. But remarkably the two equations for $G_{\mu\nu}$ and $\widetilde{G}_{\mu\nu}$ in the middle looks like free Maxwell's equations. Indeed, since $G_{\mu\nu}$ and $\widetilde{G}_{\mu\nu}$ are dual to each other, we can express $G_{\mu\nu}$ by one-form potential G_μ

$$
(41)\qquad G_{\mu\nu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu = \partial_\mu G_\nu - \partial_\nu G_\mu,
$$

using the fact $\nabla_\mu \widetilde{G}^{\mu\nu} = 0$. Equivalently, we can express $\widetilde{G}_{\mu\nu}$ by one-form potential $\tilde{G}_\mu$

$$
(42)\qquad \widetilde{G}_{\mu\nu} = \nabla_\mu \tilde{G}_\nu - \nabla_\nu \tilde{G}_\mu = \partial_\mu \tilde{G}_\nu - \partial_\nu \tilde{G}_\mu,
$$

using the fact $\nabla_\mu G^{\mu\nu} = 0$. So we can express the equations of the restricted metric $G_{\mu\nu}$ and $\widetilde{G}_{\mu\nu}$ as a Maxwell-type second-order differential equation in terms of the potential G_μ,

$$
(43)\qquad \nabla_\mu G^{\mu\nu} = 0, \qquad G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu.
$$

This is really remarkable and surprising, because this shows that the dynamical part of A_2 gravity can be described by an Abelian gauge theory.

4$\dot{}$2. B_2 *gravity.* – Now, let

$$
\begin{aligned}
(44)\qquad & \mathbf{g}_{\mu\nu} = \hat{\mathbf{g}}_{\mu\nu} + \mathbf{G}_{\mu\nu}, \\
& \hat{\mathbf{g}}_{\mu\nu} = \mathcal{J}_{\mu\nu}\mathbf{k} - \widetilde{\mathcal{J}}_{\mu\nu}\tilde{\mathbf{k}}, \qquad \mathbf{G}_{\mu\nu} = \mathcal{K}_{\mu\nu}\mathbf{j} - \widetilde{\mathcal{K}}_{\mu\nu}\tilde{\mathbf{j}} + \mathcal{L}_{\mu\nu}\mathbf{l} - \widetilde{\mathcal{L}}_{\mu\nu}\tilde{\mathbf{l}}, \\
& \mathcal{J}_{\mu\nu} = e^a_\mu e^b_\nu j_{ab}, \qquad \mathcal{K}_{\mu\nu} = e^a_\mu e^b_\nu k_{ab}, \qquad \mathcal{L}_{\mu\nu} = e^a_\mu e^b_\nu l_{ab}, \\
& \widetilde{\mathcal{J}}_{\mu\nu} = e^a_\mu e^b_\nu \tilde{j}_{ab}, \qquad \widetilde{\mathcal{K}}_{\mu\nu} = e^a_\mu e^b_\nu \tilde{k}_{ab}, \qquad \widetilde{\mathcal{L}}_{\mu\nu} = e^a_\mu e^b_\nu \tilde{l}_{ab},
\end{aligned}
$$

and find, with $\mathbf{Z}_\mu = 0$, (35) is reduced to

$$
\begin{gathered}
\mathcal{J}_{\mu\nu}(\partial^\nu K^\rho - \partial^\rho K^\nu) - \widetilde{\mathcal{J}}_{\mu\nu}(\partial^\nu \widetilde{K}^\rho - \partial^\rho \widetilde{K}^\nu) = 0, \\
\nabla_\mu \mathcal{J}^{\mu\nu} = 0, \qquad \nabla_\mu \widetilde{\mathcal{J}}^{\mu\nu} = 0, \\
\hat{\mathscr{D}}_\mu \mathbf{G}^{\mu\nu} + \mathcal{J}^{\mu\nu} \hat{D}_\mu \mathbf{k} - \widetilde{\mathcal{J}}^{\mu\nu} \hat{D}_\mu \tilde{\mathbf{k}} = 0,
\end{gathered}
\tag{45}
$$

which describes the restricted B_2 gravity.

Here again the first and last equations can be viewed as the constraint equations which determine the connection in terms of the metric. But the two equations for $\mathcal{J}_{\mu\nu}$ and $\widetilde{\mathcal{J}}_{\mu\nu}$ in the middle allows us to introduce one-form potential $\mathcal{J}_\mu$ for $\mathcal{J}_{\mu\nu}$

$$
\mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu, \tag{46}
$$

or $\widetilde{\mathcal{J}}_{\mu\nu} = \partial_\mu \widetilde{\mathcal{J}}_\nu - \partial_\nu \widetilde{\mathcal{J}}_\mu$. With this we can express the equations of the restricted metric $\mathcal{J}_{\mu\nu}$ and $\widetilde{\mathcal{J}}_{\mu\nu}$ as a Maxwell-type second-order differential equation in terms of the potential $\mathcal{J}_\mu$,

$$
\nabla_\mu \mathcal{J}^{\mu\nu} = 0, \qquad \mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu. \tag{47}
$$

This shows that the dynamical part of B_2 gravity can also be described by an Abelian gauge theory.

To avoid any misunderstanding, however, we emphasize that this Abelian structure exists only in the restricted gravity, in the absence of the valence connection. As soon as we include the valence connection we no longer have $\nabla^\mu \widetilde{G}_{\mu\nu} = 0$ or $\nabla^\mu \widetilde{\mathcal{J}}_{\mu\nu} = 0$, so that $G_{\mu\nu}$ or $\widetilde{\mathcal{J}}_{\mu\nu}$ does not admit the Abelian potential. So the Abelian structure disappears when the restricted gravity includes the valence connection, or in general a gravitational source.

Clearly both (43) and (47) imply that the dynamical field of the restricted gravity is described by a massless spin-one field. But this is the only dynamical degrees that we have, so that this must be identified as the graviton. This view is endorsed by the fact that the restricted gravity accommodates the well known gravitational plane-wave solution [19]. This means that the graviton can be described by a massless spin-one field. This is a most important outcome of our analysis.

At first thought this view sounds heretical, but actually is not so. First of all, the massless spin-one field has the right degrees of freedom for the graviton. Just as the massless spin-two metric it has two physical degrees. Besides, the metric is not the only field which describes the graviton. Classically the metric is equivalent to tetrads, so that the graviton can also be described by tetrads. And they are bi-vector, vector in both the Lorentz space and the coordinate space. So each of the four tetrads becomes a vector. Furthermore, just like the metric, our dynamical fields $G_{\mu\nu}$ and $\mathcal{J}_{\mu\nu}$ are made of tetrads. So it is really not a strange idea to describe the graviton by them. The new (and surprising) thing of our analysis is that they can be expressed by Abelian potentials, through the equation of motion. This leads us to the idea of massless spin-one graviton.

5. – Discussions

In this paper we have discussed the Abelian decomposition of Einstein's theory. Imposing proper magnetic isometries to the gravitational connection, we have shown how to decompose the gravitational connection and the curvature tensor into the restricted part of the maximal Abelian subgroup H of Lorentz group G and the valence part of G/H component which plays the role of the Lorentz covariant gravitational source of the restricted connection, without compromising the general invariance.

This tells that Einstein's theory can be viewed as a theory of the restricted gravity made of the restricted connection in which the valence connection plays the role of the gravitational source of the restricted gravity. We show that there are two different Abelian decompositions of Einstein's theory, non–light-like A_2 decomposition (the rotation/boost decomposition) and light-like B_2 decomposition (the null decomposition), because Lorentz group has two maximal Abelian subgroups.

An important ingredient of the decomposition is the concept of Lorentz covariant four-index metric tensor $\mathbf{g}_{\mu\nu}$ which replaces the role of the two-index space-time metric $g_{\mu\nu}$. We have shown that the metric-compatibility condition of the connection $\nabla_\alpha g_{\mu\nu} = 0$ is replaced by the gauge (and generally) covariant condition $\mathscr{D}_\mu \mathbf{g}^{\mu\nu} = 0$.

From theoretical point of view, the above decomposition of gravitation differs from the Abelian decomposition of non-Abelian gauge theory in one important respect. In the gauge theory the fundamental ingredient is the gauge potential, and the decomposition of the potential provides a complete decomposition of the theory. But in gravitation the fundamental field is assumed to be the metric, not the connection (the potential). Because of this the decomposition of the connection gives us the the decomposition of the metric only indirectly, through the equation of motion. It would be very interesting to see if one can actually decompose the metric explicitly, and decompose Einstein's theory in terms of the metric.

Nevertheless the above decomposition of Einstein's theory has deep implications. First of all, this tells that we can construct a restricted theory of gravitation, actually two of them, which is generally invariant (or equivalently Lorentz gauge invariant) but has fewer physical degrees of freedom than what we have in Einstein's theory. This means that we can separate the Abelian part of gravity which describes the core dynamics of Einstein's theory without compromising the general invariance.

Moreover, our analysis shows that we could describe the restricted gravity by an Abelian gauge theory with one-form potential. In other words, our result implies that the graviton can be described by a massless spin-one potential, in stead of the spin-two metric. Indeed, we can show that the B_2 gravity describes the well-known Einstein-Rosen-Bondi's gravitational plane-wave solution [19]. As importantly, we can argue that the A_2 gravity could describe Weyl's C-class space-time and the asymptotically flat radiative space-time of Bicak and Schmidt [20]. This tells that the restricted gravity is able to describe non-trivial space-times, in particular the gravitational plane wave. This has an important implication, because this confirms that the restricted gravity can indeed describe the graviton, at least classically. This tells that it could play a crucial role for us to construct the quantum gravity.

Furthermore, the decomposition makes the topology of Einstein's theory more transparent. Indeed with the Abelian decomposition we can study the topological structures of the theory more easily, because the topological characteristics are imprinted in the magnetic symmetry. For example, the A_2 decomposition makes it clear that the topology of Einstein's theory is closely related to the topology of $SU(2)$ gauge theory. This is natural, because $SU(2)$ forms a subgroup of Lorentz group. This similarity between Einstein's theory and $SU(2)$ gauge theory might be very useful for us to study the gravito-magnetic monopole in Einstein's theory which has the monopole topology $\pi_2(S^2)$ [21, 22].

This strongly implies that Einstein's theory may have the multiple vacua similar to what we find in the $SU(2)$ gauge theory. This turns out to be true. In fact with a proper magnetic isometry we can construct all possible vacuum space-times, and show that Einstein's theory has exactly the same multiple vacua that we have in the $SU(2)$ gauge theory. This is because $\pi_3(SO(3,1)) \simeq \pi_3(SU(2)) \simeq \pi_3(S^3) \simeq \pi_3(S^2)$. This tells that the vacuum space-time can be classified by the knot topology $\pi_3(S^3) \simeq \pi_3(S^2)$ [23].

This could have a far-reaching consequence. Just as in $SU(2)$ gauge theory, the multiple vacua in Einstein's theory can be unstable against quantum fluctuation. And there is a real possibility that Einstein's theory may admit the gravito-instantons which can connect topologically distinct vacua and thus allow the vacuum tunneling [23, 24]. Clearly this will have an important implication in quantum gravity.

The details of the subject with interesting applications will be discussed separately [25].

* * *

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